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ON A NEW PARTICULAR SOLUTION OF THE EQUATIONS OF MOTION OF A HEAVY SOLID IN A LIQUID*

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The Kirchhoff-Clebsch equations of motion of a free heavy solid in a fluid are considered under Chaplygin conditions /1/. The precession motions of a solid are investigated in which the rotating part consists of two rotary motions, the direction of rotation of the first is constant in space, and the second is constant in the solid, the angle between the two being constant. Means for finding precession motions of the solid in fluid are indicated. A particular solution is given for the case when the angular velocity components of the rotary motions are equal and the direction of their rotation mutually perpendicular. This solution is similar to that of Grioli problem of the motion of a heavy solid with a single fixed point. The geometric interpretation of a solid in fluid, defined by that solution is given.

1. Consider the motion of a free solid bounded by a simply connected surface in a homogenous gravitational field of force, unbounded in all directions, in a homogeneous incompressible ideal fluid. The fluid performs irrotational motion, and is at rest at infinity. We will assume that there is a statically and dynamically balanced rotor in the solid that rotates at consant relative angular velocity about an axis permanently fixed to the solid.

Let the weight of the fluid displaced by the solid be equal to the combined weights of the solid and rotor. We denote by R_i, P_i, λ_i $(i = 1, 2, 3)$ the projections on the axes of coordinates Ox_1x_2 permanently fixed to the solid of the vector $\mathbf{R}$ of the angular momentum of the system of gyrostat plus liquid (impulse force /1-3/), by the vector $\mathbf{P}$ its momentum relative to the point O (impulse couple /1-3/), and by the vector λ of the gyrostatic moment of the rotor. The kinetic energy of the system has the form /1-3/

$$T = \frac{1}{2}\sum_{i,j=1}^{3}(a_{ij}P_iP_j + b_{ij}R_iR_j + 2c_{ij}P_iR_j), \quad a_{ij} = a_{ji}, \quad b_{ij} = b_{ji} \tag{1.1}$$

where a_{ij}, b_{ij}, c_{ij} are constants defined for the particular system. The projections of u_i, Ω_i $(i = 1, 2, 3)$ on x_i axes of the vectors of the translational angular velocity $\mathbf{u}$ and the instantaneous angular velocity Ω of the solid are determined by the formulae

$$u_i = \partial T/\partial R_i, \quad \Omega_i = \partial T/\partial P_i \quad (i = 1, 2, 3). \tag{1.2}$$

Assuming the impulse force $\mathbf{R}$ $(\mathbf{R}^2 = H^2 = \text{const})$ to be directed along an ascending vertical line, we have the following equations of motion of the system /1-3/:

$$\begin{aligned} &dR_1/dt + \Omega_2R_3 - \Omega_3R_2 = 0 && (123) \\ &dP_1/dt + \Omega_2(P_3 + \lambda_3) - \Omega_3(P_2 + \lambda_2) + u_2R_3 - u_3R_2 = \\ &\quad e_2R_3 - e_3R_2 && (123) \end{aligned} \tag{1.3}$$

where e_1, e_2, e_3 are constant projections on the x_i axes of the radius vector drawn from the centre of mass of the volume bounded by the outer surface of the solid to the centre of mass of the gyrostat.

The inertial motion of the solid, bounded by a multiply connected surface, in an unbounded fluid, is also defined by (1.3) /4/.

2. We shall call the motion of a solid precessional, if its rotational part is composed of two rotary motions, the first of which is fixed in space and the second is fixed in the solid, and the angle between these directions is constant.

We shall describe a method of determining the precessional motion of a solid in a fluid, when the components of the rotary motion are constant.

We denote by γ the unit vector of the direction of the rotary motion fixed in space. We assume without loss of generality that the rotary motion whose direction is fixed in the solid takes place about the x_3 axis with the unit vector $\mathbf{i}_3$. The unit vector of the pulsed force $\mathbf{R} = H\mathbf{v}$ fixed in space is denoted by $\mathbf{v}$. The vectors γ and $\mathbf{v}$ satisfy the equations

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$$d\gamma/dt = \gamma \times \Omega, \quad d\mathbf{v}/dt = \mathbf{v} \times \Omega \tag{2.1}$$

$$\mathbf{v}^2 = 1, \quad \gamma^2 = 1, \quad \mathbf{v}\cdot\gamma = \cos\varkappa = \text{const}, \quad \gamma\cdot\mathbf{i}_3 = \cos\theta = \text{const}. \tag{2.2}$$

Multiplying the first equation of (2.1) scalarly by $\mathbf{i}_3$ and taking into account the last of conditions (2.2), we obtain the relation $\mathbf{i}_3\cdot(\gamma \times \Omega) = 0$. This implies that the vector Ω can be represented in the form

$$\Omega = \varphi^{\cdot}\mathbf{i}_3 + \psi^{\cdot}\gamma, \quad \varphi^{\cdot} = \text{const}, \quad \psi^{\cdot} = \text{const}. \tag{2.3}$$

Substituting (2.3) into (2.1), we obtain the following equations for γ and $\mathbf{v}$;

$$\gamma^{\cdot} = \varphi^{\cdot}(\gamma \times \mathbf{i}_3), \quad \mathbf{v}^{\cdot} = \varphi^{\cdot}(\mathbf{v} \times \mathbf{i}_3) + \psi^{\cdot}(\mathbf{v} \times \gamma). \tag{2.4}$$

Integrating the first of these equations, taking (2.2) into account, we obtain for the projections γ_i of vector γ on the x_i axis the expressions

$$\gamma_1 = \sin\theta\sin\varphi, \quad \gamma_2 = \sin\theta\cos\varphi, \quad \gamma_3 = \cos\theta, \quad \varphi = \varphi^{\cdot}t + \varphi_0. \tag{2.5}$$

To determine vector $\mathbf{v}$ that satisfies the second of equations (2.4) and relations (2.2), we use the equation

$$\mathbf{v} = \left(\cos\varkappa + \frac{\cos\theta\sin\varkappa\sin\psi}{\sin\theta}\right)\gamma - \frac{\sin\varkappa\sin\psi}{\sin\theta}\mathbf{i}_3 - \frac{\sin\varkappa\cos\psi}{\sin\theta}(\gamma \times \mathbf{i}_3) \tag{2.6}$$

where $\psi = \psi^{\cdot}t + \psi_0$ and ψ_0 is an arbitrary constant.

From (2.3), (2.5) and (2.6) and the equation $\mathbf{R} = H\mathbf{v}$ we have

$$\Omega_1 = \psi^{\cdot}\sin\theta\sin\varphi, \quad \Omega_2 = \psi^{\cdot}\sin\theta\cos\varphi, \quad \Omega_3 = \psi^{\cdot}\cos\theta + \varphi^{\cdot} \tag{2.7}$$

$$\begin{aligned} R_1 &= H[\cos\varkappa\sin\theta\sin\varphi + \sin\varkappa(\cos\theta\sin\psi\sin\varphi - \\ &\quad \cos\psi\cos\varphi)] \\ R_2 &= H[\cos\varkappa\sin\theta\cos\varphi + \sin\varkappa(\cos\theta\sin\psi\cos\varphi + \\ &\quad \cos\psi\sin\varphi)] \\ R_3 &= H(\cos\varkappa\cos\theta - \sin\varkappa\sin\theta\sin\psi). \end{aligned} \tag{2.8}$$

Having determined Ω and $\mathbf{R}$, we obtain the vectors $\mathbf{P}$ and $\mathbf{U}$ from (1.2) and represent them in the tensor form

$$\begin{aligned} &\mathbf{P} = \mathbf{A}\cdot\Omega - \mathbf{C}\cdot\mathbf{R}, \quad \mathbf{U} = \mathbf{C}^{\tau}\cdot\Omega + \mathbf{B}\cdot\mathbf{R} \\ &\mathbf{A} = \| A_{ij} \|_1^3 = a^{-1}, \quad \mathbf{C} = \| C_{ij} \|_1^3 = a^{-1}\cdot c, \quad \mathbf{B} = \| B_{ij} \|_1^3 = \\ &b - c^{\tau}\cdot a^{-1}\cdot c \\ &a = \| a_{ij} \|_1^3, \ b = \| b_{ij} \|_1^3, \quad c = \| c_{ij} \|_1^3. \end{aligned} \tag{2.9}$$

From (2.9), the equation $\mathbf{R} = H\mathbf{v}$, and (2.3) and (2.6), we obtain

$$\begin{aligned} \mathbf{P} &= \left[\psi^{\cdot}\mathbf{A} - H\left(\cos\varkappa + \frac{\sin\varkappa\cos\theta\sin\psi}{\sin\theta}\right)\mathbf{C}\right]\cdot\gamma + \\ &\quad \left[\varphi^{\cdot}\mathbf{A} + H\frac{\sin\varkappa\sin\psi}{\sin\theta}\mathbf{C}\right]\cdot\mathbf{i}_3 + H\frac{\sin\varkappa\cos\psi}{\sin\theta}\mathbf{C}\cdot(\gamma \times \mathbf{i}_3) \\ u &= \left[\psi^{\cdot}\mathbf{C}^{\tau} + H\left(\cos\varkappa + \frac{\sin\varkappa\cos\theta\sin\psi}{\sin\theta}\right)\mathbf{B}\right]\cdot\gamma + \\ &\quad \left[\varphi^{\cdot}\mathbf{C}^{\tau} - H\frac{\sin\varkappa\sin\psi}{\sin\theta}\mathbf{B}\right]\cdot\mathbf{i}_3 - H\frac{\sin\varkappa\cos\psi}{\sin\theta}\mathbf{B}\cdot(\gamma \times \mathbf{i}_3). \end{aligned}$$

From this we obtain for P_i, u_i the formulae

$$\begin{aligned} P_1 &= (A_{11}\sin\varphi + A_{12}\cos\varphi)\psi^{\cdot}\sin\theta + A_{13}(\varphi^{\cdot} + \psi^{\cdot}\cos\theta) - \\ &\quad C_{11}H[\cos\varkappa\sin\theta\sin\varphi + (\cos\theta\sin\varphi\sin\psi - \cos\varphi\cos\psi)\sin\varkappa] - \\ &\quad C_{12}H[\sin\varkappa(\cos\theta\cos\varphi\sin\psi + \sin\varphi\cos\psi) + \\ &\quad \cos\varkappa\sin\theta\cos\varphi] + C_{13}H(\sin\varkappa\sin\theta\sin\psi - \cos\varkappa\cos\theta) \\ &\qquad (123) \end{aligned} \tag{2.10}$$

$$\begin{aligned} u_1 &= (C_{11}\sin\varphi + C_{21}\cos\varphi)\psi^{\cdot}\sin\theta + C_{31}(\varphi^{\cdot} + \psi^{\cdot}\cos\theta) + \\ &\quad B_{11}H[\cos\varkappa\sin\theta\sin\varphi + (\cos\theta\sin\varphi\sin\psi - \cos\varphi\cos\psi)\cdot \\ &\quad \cdot\sin\varkappa] + B_{12}H[\sin\varkappa(\cos\theta\cos\varphi\sin\psi + \\ &\quad \sin\varphi\cos\psi) + \cos\varkappa\sin\theta\cos\varphi] + B_{13}H(\cos\varkappa\cos\theta - \\ &\quad \sin\varkappa\sin\theta\sin\psi) \qquad (123)' \end{aligned} \tag{2.11}$$

where the symbol (123) indicates that two other formulae are obtained from this one by permuting the indices 1, 2, 3 of P_i, u_i, and the first index of the constants A_{ij}, B_{ij}, C_{ij}. The prime on this symbol indicates that in (2.11) for C_{ij} the second index and not the first is to be changed according to permutation (123).

Substituting (2.7), (2.8), (2.10) and (2.11) for Ω_i, R_i, P_i, u_i into the second group of (1.3) and stipulating that they are identically satisfied by φ and ψ, we obtain the required conditions for precessional motions of the solid in the fluid to exist.

3. Let

$$\theta = \pi/2, \quad \varphi^{\cdot} = \psi^{\cdot} = \text{const}, \quad \varphi = \psi = \varphi^{\cdot} t + \varphi_0. \tag{3.1}$$

In this case the conditions for precessional motions to exist have the form

$$\begin{aligned}
&B_{ij} H \sin\varkappa = 0, \quad (B_{22} - B_{11}) H \sin\varkappa = 0 \quad (i, j = 1, 2, 3; \ i \neq j) \\
&[(C_{12} + C_{21})\varphi^{\cdot} + {}^1/_2 (4B_{12}\cos\varkappa - B_{23}\sin\varkappa) H] H \sin\varkappa = 0 \\
&[(C_{23} + C_{32})\varphi^{\cdot} + (B_{12}\sin\varkappa + 2B_{23}\cos\varkappa) H] H \sin\varkappa = 0 \\
&\{(C_{22} - C_{11})\varphi^{\cdot} + [(B_{22} - B_{11})\cos\varkappa + \\
&\quad {}^1/_2 B_{13}\sin\varkappa] H\} H \sin\varkappa = 0 \\
&8A_{12}\varphi^{\cdot 2} + [(C_{23} + C_{32})\sin\varkappa - 4(C_{12} + C_{21})\cos\varkappa]\varphi^{\cdot} H + \\
&\quad (B_{12}\sin\varkappa - 2B_{23}\cos\varkappa) H \sin\varkappa = 0 \\
&A_{23}\varphi^{\cdot 2} - [(C_{12} + C_{21})\sin\varkappa + ((C_{23} + C_{32})\cos\varkappa]\varphi^{\cdot} H + \\
&\quad (B_{12}\sin\varkappa\cos\varkappa - B_{23}) H = 0 \\
&[(C_{23} + C_{32})\varphi^{\cdot} + {}^1/_2 (2B_{23}\cos\varkappa - B_{12}\sin\varkappa) H] H \sin\varkappa = \\
&\quad e_2 H \sin\varkappa, \ [(C_{12} + C_{21})\varphi^{\cdot} + {}^1/_2 (4B_{12}\cos\varkappa - \\
&\quad B_{23}\sin\varkappa) H] H \sin\varkappa = -2e_2 H\cos\varkappa - 2\lambda_2\varphi^{\cdot}
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&[(C_{23} + C_{32})\varphi^{\cdot} - (B_{12}\sin\varkappa - 2B_{23}\cos\varkappa) H] H \sin\varkappa = 0 \\
&A_{23}\varphi^{\cdot 2} + [(C_{12} + C_{21})\sin\varkappa - (C_{23} + C_{32})\cos\varkappa]\varphi^{\cdot} H - \\
&\quad {}^1/_2 [B_{12}\sin 2\varkappa + B_{23}(2 - 3\sin^2\varkappa)] H = 0
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&A_{23}\varphi^{\cdot 2} + [(C_{12} + C_{21})\sin\varkappa + (C_{23} + C_{32})\cos\varkappa]\varphi^{\cdot} H + \\
&\quad {}^1/_4 [2B_{12}\sin 2\varkappa + B_{23}(4 - 7\sin^2\varkappa)] H = -2\lambda_2\varphi^{\cdot}. \\
&2A_{12}\varphi^{\cdot 2} - [(C_{12} + C_{21})\cos\varkappa - {}^5/_4 (C_{23} + C_{32})\sin\varkappa]\varphi^{\cdot} H - \\
&\quad {}^1/_4 [B_{12}\sin\varkappa - 2B_{23}\cos\varkappa] H^2 \sin\varkappa = e_2 H \sin\varkappa \\
&2A_{12}\varphi^{\cdot 2} - [2(C_{12} + C_{21})\cos\varkappa - (C_{23} + C_{32})\sin\varkappa]\varphi^{\cdot} H - \\
&\quad [B_{12}(2 - 3\sin^2\varkappa) - B_{23}\sin\varkappa\cos\varkappa] H^2 = -e_2 H \sin\varkappa
\end{aligned}$$

$$\begin{aligned}
&\{(C_{13} + C_{31})\varphi^{\cdot} + [(B_{11} - B_{33})\sin\varkappa + 2B_{13}\cos\varkappa] H\} H \sin\varkappa = 0 \\
&[(C_{13} + C_{31})\varphi^{\cdot} + (B_{22} - B_{33}) H \sin\varkappa] H \sin\varkappa = 0 \\
&\{(2C_{33} + C_{22} - C_{11})\varphi^{\cdot} + [(B_{22} - B_{11})\cos\varkappa - \\
&\quad {}^1/_2 B_{13}\sin\varkappa] H\} H \sin\varkappa = 2e_1 H \cos\varkappa + 2\lambda_1\varphi^{\cdot} \\
&(A_{22} - A_{11})\varphi^{\cdot 2} - [(C_{13} + C_{31})\sin\varkappa + 2(C_{22} - C_{11})\cos\varkappa]\varphi^{\cdot} H + \\
&\quad {}^1/_2 [(B_{11} - B_{22})(2 - 3\sin^2\varkappa) - B_{13}\sin 2\varkappa] H^2 = e_1 H \sin\varkappa \\
&(A_{11} - A_{22} - A_{33})\varphi^{\cdot 2} + [(C_{33} + C_{22} - C_{11})\cos\varkappa + {}^3/_4 (C_{13} + \\
&\quad C_{31})\sin\varkappa]\varphi^{\cdot} H + {}^1/_4 [(B_{33} - B_{11})\sin\varkappa - 2B_{13}\cos\varkappa] H^2 \sin\varkappa = \\
&\quad H(e_1\sin\varkappa + e_3\cos\varkappa) + \lambda_3\varphi^{\cdot} \\
&(A_{11} - A_{22} + A_{33})\varphi^{\cdot 2} - [(C_{33} + C_{11} - C_{22})\cos\varkappa - {}^3/_4 (C_{13} + \\
&\quad C_{31})\sin\varkappa]\varphi^{\cdot} H + {}^1/_4 (B_{33} - B_{22}) H^2 \sin^2\varkappa = -e_3 H \cos\varkappa - \\
&\quad \lambda_3\varphi^{\cdot} \\
&A_{13}\varphi^{\cdot 2} - [2C_{22}\sin\varkappa + (C_{13} + C_{31})\cos\varkappa]\varphi^{\cdot} H + {}^1/_2 [(B_{33} - \\
&\quad B_{11})\sin 2\varkappa + 2B_{13}(2 - 3\cos^2\varkappa)] H^2 = -e_3 H \sin\varkappa \\
&A_{13}\varphi^{\cdot 2} + [2(C_{11} - C_{33})\sin\varkappa + (C_{13} + C_{31})\cos\varkappa]\varphi^{\cdot} H + \\
&\quad [(B_{11} - B_{33})\sin\varkappa\cos\varkappa + {}^1/_4 B_{13}(4 - 5\sin^2\varkappa)] H^2 = \\
&\quad -e_3 H \sin\varkappa - 2\lambda_1\varphi^{\cdot} \\
&A_{13}\varphi^{\cdot 2} + [2C_{11}\sin\varkappa - (C_{13} + C_{31})\cos\varkappa]\varphi^{\cdot} H + \\
&\quad {}^1/_2 [(B_{33} - B_{22})\sin 2\varkappa + B_{13}(1 - 3\cos^2\varkappa)] H^2 = -e_3 H \sin\varkappa.
\end{aligned} \tag{3.4}$$

From (3.2) we successively have

$$\begin{aligned}
&B_{12} = B_{13} = B_{23} = 0, \quad B_{11} = B_{22} \\
&C_{12} + C_{21} = 0, \quad C_{23} + C_{32} = 0, \quad C_{11} = C_{22}, \quad A_{12} = A_{23} = 0 \\
&e_2 = 0, \quad \lambda_2 = 0.
\end{aligned} \tag{3.5}$$

When conditions (3.5) are satisfied equations (3.3) are identically satisfied.

Let us now consider (3.4), taking (3.5) into account. Adding the fifth and sixth equations term by term and taking into consideration the first and fourth, we obtain

$$e_1 = 0. \tag{3.6}$$

The last two equations are compatible only when the condition

$$C_{11} = 0 \tag{3.7}$$

is satisfied. When conditions (3.6) and (3.7) are satisfied, the seventh equation, which is the corollary of the first, third and eighth, and the first two equations are the same. Now when conditions (3.5)–(3.7) are satisfied, (3.4) reduces to the following equations:

$$\lambda_1 = C_{33} H \sin \varkappa \tag{3.8}$$

$$(C_{13} + C_{31})\, \varphi^{\cdot} = (B_{33} - B_{11})\, H \sin \varkappa, \ (A_{22} - A_{11})\, \varphi^{\cdot} = (C_{13} + C_{31})\, H \sin \varkappa \tag{3.9}$$

$$(C_{33}\varphi^{\cdot} - e_3)\, H \sin \varkappa = (A_{13}\varphi^{\cdot} + \lambda_1)\, \varphi^{\cdot}, \ (C_{33}\varphi^{\cdot} - e_3)\, H \cos \varkappa = (A_{33}\varphi^{\cdot} + \lambda_3)\, \varphi^{\cdot}. \tag{3.10}$$

From (3.8) and (3.10) we have

$$[(A_{13}\varphi^{\cdot} + \lambda_1)^2 + (A_{33}\varphi^{\cdot} + \lambda_3)^2]\, A_{13}^2 \varphi^{\cdot 4} = p^2\, (A_{13}\varphi^{\cdot} + \lambda_1)^2 \tag{3.11}$$

$$\operatorname{tg} \varkappa = \frac{A_{13}\varphi^{\cdot} + \lambda_1}{A_{33}\varphi^{\cdot} + \lambda_3}, \quad H = \frac{\lambda_1}{C_{33} \sin \varkappa} \tag{3.12}$$

where $p = e_3 H$ is the product of themass of the fluid displaced by the solid and the distance from the centre of mass of the gyrostat to the centre of gravity of the volume bounded by the outer surface of the solid.

From (3.9) we have

$$(C_{13} + C_{31})^2 = (A_{22} - A_{11})(B_{33} - B_{11}) \tag{3.13}$$

$$\varphi^{\cdot} = \frac{\lambda_1 (C_{13} + C_{31})}{C_{33} (A_{22} - A_{11})}\,. \tag{3.14}$$

Substituting (3.14) into (3.11) we obtain

$$[C_{33}(A_{22} - A_{11}) + A_{13}(C_{13} + C_{31})]^2\, C_{33}^4 (A_{22} - A_{11})^4 p^2 = \{[C_{33}(A_{22} - A_{11}) + A_{13}(C_{13} + C_{31})]^2 \lambda_1^2 + [A_{33}(C_{13} + C_{31})\lambda_1 + C_{33}(A_{22} - A_{11})\lambda_3]^2\}\, A_{13}^2 (C_{13} + C_{31})^4 \lambda_1^2\,. \tag{3.15}$$

When conditions (3.1), (3.5)–(3.7) are satisfied, formulae (2.8), (2.10), (2.7), (2.11) and (2.5) take the form

$$\begin{aligned}
R_1 &= H(\cos \varkappa \sin \varphi - \sin \varkappa \cos^2 \varphi), \quad R_3 = -H \sin \varkappa \sin \varphi \\
R_2 &= H(\cos \varkappa + \sin \varkappa \sin \varphi)\cos \varphi, \quad \varphi = \varphi^{\cdot} t + \varphi_0 \\
P_1 &= (A_{13} + A_{11} \sin \varphi)\, \varphi^{\cdot} - [C_{12}(\sin \varkappa \sin \varphi + \cos \varkappa)\cos \varphi - C_{13} \sin \varkappa \sin \varphi]\, H \\
P_2 &= A_{22}\, \varphi^{\cdot} \cos \varphi + [C_{12}(\cos \varkappa \sin \varphi - \sin \varkappa \cos^2 \varphi) + C_{23} \sin \varkappa \sin \varphi]\, H \\
P_3 &= (A_{33} + A_{13} \sin \varphi)\, \varphi^{\cdot} + [C_{31}(\sin \varkappa \cos^2 \varphi - \cos \varkappa \sin \varphi) + C_{23}(\sin \varkappa \sin \varphi + \cos \varkappa)\cos \varphi + C_{33} \sin \varkappa \sin \varphi]\, H
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
\Omega_1 &= \varphi^{\cdot} \sin \varphi, \quad \Omega_2 = \varphi^{\cdot} \cos \varphi, \quad \Omega_3 = \varphi^{\cdot} \\
u_1 &= (C_{31} - C_{12} \cos \varphi)\, \varphi^{\cdot} + B_{11} H (\cos \varkappa \sin \varphi - \sin \varkappa \cos^2 \varphi) \\
u_2 &= (-C_{23} + C_{12} \sin \varphi)\, \varphi^{\cdot} - B_{11} H (\sin \varkappa \sin \varphi - \cos \varkappa) \cos \varphi \\
u_3 &= (C_{33} + C_{13} \sin \varphi + C_{23} \cos \varphi)\, \varphi^{\cdot} - B_{33} H \sin \varkappa \sin \varphi
\end{aligned} \tag{3.17}$$

$$\gamma_1 = \sin \varphi, \quad \gamma_2 = \cos \varphi, \quad \gamma_3 = 0\,. \tag{3.18}$$

The expression for the system kinetic energy, taking (1.1), (1.2), (2.9), and (3.5) into account can be represented in the form /2, 5/

$$2T = \mathbf{\Omega} \cdot \mathbf{P} + \mathbf{u} \cdot \mathbf{R} = \mathbf{\Omega} \cdot A \cdot \mathbf{\Omega} + \mathbf{R} \cdot B \cdot \mathbf{R} = A_{11}\Omega_1^2 + A_{22}\Omega_2^2 + A_{33}\Omega_3^2 + 2A_{13}\Omega_1\Omega_3 + B_{11}(R_1^2 + R_2^2) + B_{33} R_3^2\,. \tag{3.19}$$

Let us restate the results obtained so far. If the kinetic energy of the system of gyrostat and fluid has the form (3.19) and conditions (3.5) and (3.6) are satisfied, equations (1.3) have the solution (3.16), (3.17) in which the constant $\varphi^{\cdot}$ is determined by (3.11), the constants $\varkappa$ and H are calculated using formulae (3.12), and the parameters $C_{13}, C_{31}, C_{33}, A_{11}, A_{22}, A_{33}, A_{13}, B_{11}, B_{33}, \lambda_1, \lambda_3, p$ are connected by the relations (3.13) and (3.15). This solution defines the motion of the solid whose rotary part is in regular precession about an axis fixed in space. The projections of the unit vector γ of that axis are determined by (3.18), which makes an angle $\varkappa$ with the ascending vertical line, and the spin occurs about the axis which is parallel to the x_3 axis and orthogonal to the vector γ.

When $\lambda_1 = C_{12} = C_{23} = C_{13} = C_{31} = C_{33} = A_{11} - A_{22} = B_{11} = B_{33} = 0$ this solution becomes the Kharlamova solution /6/ for the problem of a heavy gyrostat with a single fixed point and, when $\lambda_3 = 0$, the last one becomes the Grioli solution /7/.

Remark. In the solution indicated here, we can, without loss of generality, put $B_{11} = 0$. Indeed, (1.3) do not change, if in (1.1) the coefficients $b_{ii} - b$ are substituted for b_{ii} ($i =$

1, 2, 3), when the constants B_{ii}, by virtue of (2.9), take the value $B_{ii} - b$. Setting $b = B_{ii}$, we obtain the required result. Mechanically this means that in the motion of the solid considered here, the constant translational motion with velocity $\mathbf{U} = b\mathbf{R}$ in the direction of the vertical line is rejected.

4. We shall now give a full geometric interpretation of the motion of the solid in the fluid, as defined by the solution indicated for $B_{11} = 0$. The use of the apparatus of the cross-product calculus is convenient for this /8/.

We denote by $\mathbf{V}$ the kinematic helix defined by the formula (3.17), and represent it in the dual form $\mathbf{V} = \mathbf{\Omega} + \omega \mathbf{U}$, where ω ($\omega^2 = 0$) is the Clifford number. For the dual modulus /8/ V of the helix $\mathbf{V}$ we have the expression $V = \varphi^{\cdot} \sqrt{2}\ (1 + {}^1/_2 C_{33}\omega)$.

Let $\mathbf{\Gamma} = \gamma + \omega\gamma^{\circ}$ be the unit helix of the straight line fixed in space and having the direction of the vector γ. For $\mathbf{\Gamma}$ we have the equation

$$\mathbf{\Gamma}^{\cdot} + \mathbf{V} \times \mathbf{\Gamma} = 0, \quad \mathbf{\Gamma}^2 = 1 .$$

Separating here the moment part, for the determination of the vector γ° with projection $\gamma_1^{\circ}, \gamma_2^{\circ}, \gamma_3^{\circ}$, we obtain the equation

$$\boldsymbol{\gamma}^{\cdot\circ} + \mathbf{\Omega} \times \boldsymbol{\gamma}^{\circ} + \mathbf{u} \times \boldsymbol{\gamma} = 0, \quad \boldsymbol{\gamma} \cdot \boldsymbol{\gamma}^{\circ} = 0 .$$

Taking into account (3.17) and (3.18), we obtain

$$\begin{aligned} \gamma_1^{\circ} &= {}^1/_4 [C_{31} + \mu_1 - 4C_{12}\cos\varphi + (C_{23} - \mu_2)\sin 2\varphi + (C_{31} + \\ &\quad \mu_1)\cos 2\varphi] \\ \gamma_2^{\circ} &= {}^1/_4 [-C_{23} + \mu_2 + 4C_{12}\sin\varphi - (C_{31} - \mu_1)\sin 2\varphi + \\ &\quad (C_{23} - \mu_2)\cos 2\varphi] \\ \gamma_3^{\circ} &= {}^1/_2 [\,2C_{33} - (C_{31} - \mu_1)\sin\varphi + (C_{23} + \mu_2)\cos\varphi], \\ \varphi &= \varphi^{\cdot} t + \varphi_0 \end{aligned} \tag{4.1}$$

where μ_1, μ_2 are arbitrary constants.

We denote by $A = \alpha + \omega\alpha^{\circ}$ the dual angle /8/ between the axes of the helices $\mathbf{V}$ and $\mathbf{\Gamma}$, where α is the angle between the vectors $\mathbf{\Omega}$ and γ, and α° is the distance between the axes of the helices $\mathbf{V}$ and $\mathbf{\Gamma}$. Let us compose the scalar product of the helices $\mathbf{V}$ and $\mathbf{\Gamma}$

$$\begin{aligned} &\mathbf{V} \cdot \mathbf{\Gamma} = V\cos A, \quad \mathbf{V} \cdot \mathbf{\Gamma} = \mathbf{\Omega} \cdot \gamma + \omega(\mathbf{\Omega} \cdot \gamma^{\circ} + \mathbf{u} \cdot \gamma) \\ &V\cos A = \varphi^{\cdot} \sqrt{2}\ (1 - {}^1/_2 C_{33}\omega)(\cos\alpha - \omega\alpha^{\circ}\sin\alpha) . \end{aligned} \tag{4.2}$$

Separating in (4.2) the principal and moment parts, we have for the determination of α and α° the equations

$$\mathbf{\Omega} \cdot \gamma = \varphi^{\cdot} \sqrt{2}\cos\alpha, \quad \mathbf{\Omega} \cdot \gamma^{\circ} + \mathbf{u} \cdot \gamma = \varphi^{\cdot} \sqrt{2}\ ({}^1/_2 C_{33}\cos\alpha - \alpha^{\circ}\sin\alpha)$$

or taking (3.17), (3.18), (3.9) and (4.1) into account

$$\alpha = \frac{\pi}{4}, \quad \alpha^{\circ} = -\frac{1}{2}[C_{33} - (C_{31} + \mu_1)\sin\varphi - (C_{23} - \mu_2)\cos\varphi] .$$

Setting $\mu_1 = -C_{31}$ and $\mu_2 = C_{23}$ we finally obtain

$$\alpha = {}^1/_4\pi, \quad \alpha^{\circ} = -{}^1/_2 C_{33} . \tag{4.3}$$

Let us take any point M of the solid defined by the radius vector $\mathbf{r}(x_1, x_2, x_3)$ and denote by $\mathbf{E} = \mathbf{i}_3 + \omega\mathbf{i}_3^{\circ}$, $\mathbf{i}_3^{\circ} = \mathbf{r} \times \mathbf{i}_3$ the unit helix of the straight line which passes through the point M and is parallel to the x_3 axis. We further denote by $B = \beta + \omega\beta^{\circ}$ the dual angle between the axes of the helicies $\mathbf{V}$ and $\mathbf{E}$, and compose the scalar product of the helices $\mathbf{V}$ and $\mathbf{E}$

$$\begin{aligned} &\mathbf{V} \cdot \mathbf{E} = V\cos B, \quad \mathbf{V} \cdot \mathbf{E} = \mathbf{\Omega} \cdot \mathbf{i}_3 + \omega(\mathbf{\Omega} \cdot \mathbf{i}_3^{\circ} + \mathbf{u} \cdot \mathbf{i}_3) \\ &V\cos B = \varphi^{\cdot} \sqrt{2}\ (1 + {}^1/_2 C_{33}\omega)(\cos\beta - \omega\beta^{\circ}\sin\beta) \end{aligned} \tag{4.4}$$

Separating in (4.4) the principal and the moment parts, we obtain for the determination of β and β° the equations

$$\mathbf{\Omega} \cdot \mathbf{i}_3 = \varphi^{\cdot} \sqrt{2}\cos\beta, \quad \mathbf{\Omega} \cdot \mathbf{i}_3^{\circ} + u \cdot i_3 = \varphi^{\cdot} \sqrt{2}\ ({}^1/_2 C_{33}\cos\beta - \beta^{\circ}\sin\beta)$$

or taking (3.17) and (3.9) into account

$$\begin{aligned} &\varphi^{\cdot} = \varphi^{\cdot} \sqrt{2}\cos\beta, \quad \varphi^{\cdot}[C_{33} + (x_2 - C_{31})\sin\varphi - (x_1 - \\ &\quad C_{23})\cos\varphi] = \varphi^{\cdot}({}^1/_2 C_{33} - \beta^{\circ}) \end{aligned}$$

From this we have

$$\beta = {}^1/_4\pi, \quad \beta^\circ = -{}^1/_2\,[C_{33} + (x_2 - C_{31})\sin\varphi - (x_1 - C_{23})\cos\varphi].$$

Setting $x_1 = C_{23}$, $x_2 = C_{31}$, we finally obtain

$$\beta = {}^1/_4\pi, \quad \beta^\circ = -{}^1/_2 C_{33}. \tag{4.5}$$

By virtue of (4.2) and (4.4) the helix V can be represented in the form of the geometric sum of two helices

$$\mathbf{V} = V\,(\mathbf{\Gamma}\cos A + \mathbf{E}\cos B) \tag{4.6}$$

the dual moduli $V\cos A$ and $V\cos B$ of which are constant.

From (4.6), (4.3), and (4.5) we see that the motion of such a solid consists of two helical motions with unit helices $\mathbf{\Gamma}$ and $\mathbf{E}$ and constant dual moduli. The axes of these two helices are orthogonal to each other and the distance between the two is constant and equal to C_{33}. The helix $\mathbf{V}$ is at constant dual angles to the helices $\mathbf{\Gamma}$ and $\mathbf{E}$, hence, when the solid moves, the axis of the helix V describes in space and in the solid the same one-sheet hyperboloids, whose axes of symmetry are the axes of the helices $\mathbf{\Gamma}$ and $\mathbf{E}$.

We denote by L and N the points of intersection of the axis of helix $\mathbf{\Gamma}\times\mathbf{E}$ with the axes of the helices $\mathbf{\Gamma}$ and $\mathbf{E}$, respectively. The coordinates of the point N are $x_1 = C_{23}$, $x_2 = C_{31}$, $x_3 = C_{12}$. We introduce the fixed system of coordinates $Ly_1y_2y_3$ whose y_3 axis coincides with the axis of the helix $\mathbf{\Gamma}$. The one-sheet hyperboloids are defined by the equations.

$$y_1^2 + y_2^2 = {}^1/_4 C_{33}^2 + y_3^2 \tag{4.7}$$

$$(x_1 - C_{23})^2 + (x_2 - C_{31})^2 = {}^1/_4 C_{33}^2 + (x_3 - C_{12})^2. \tag{4.8}$$

The motion of a solid in a liquid defined by the solution (3.16), (3.17) may be represented as the result of rolling a one-sheet hyperboloid (4.8) fixed to the solid over an identical fixed hyperboloid (4.7) around a common generatrix at a constant angular velocity $\varphi^{\cdot}\sqrt{2}$, and its sliding along that generatrix at constant velocity ${}^1/_2\sqrt{2}C_{33}\varphi^{\cdot}$.

As $C_{33}\to 0$, the hyperboloids (4.7) and (4.8) in the limit transfer into their own asymptotic cones with a common vertex at the point $N\,(C_{23}, C_{31}, C_{12})$. The motion of the solid occurs so that the cone attached to it rolls over the fixed cone at constant angular velocity $\varphi^{\cdot}\sqrt{2}$.

A similar geometric representation of the motion occurs in another limit case, when the density of the liquid approaches zero. The problem of the motion of a heavy solid in a fluid then becomes the problem of the motion of a gyrosat with a single fixed point, and its solution reduces to Kharlamova's $(\lambda_3 \neq 0)$, or Grioli's $(\lambda_3 = 0)$.

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